A STUDY OF MULTIPLIER METHOD FOR MULTIPLE PARAMETERS

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ABSTRACT

The aim of this paper is to analyze multiple parameters, under the sets of assumptions on these parameters. We are able to address the two important theorems related to the proximal point algorithm that of strong convergence and that of acceptable errors with Yao and Noor's algorithm.

KEYWORDS: Proximal Point Algorithm, Convex Function, Optimization, Nonlinear Programming

INTRODUCTION

Generalized Proximal Point Algorithm

A primal–dual decomposition method is presented to solve the separable convex programming problem. Convergence to a solution and Lagrange multiplier vector occurs from an arbitrary starting point. The method is equivalent to the proximal point algorithm applied to a certain maximal monotone multifunction. In the nonseparable case, it specializes to a known method, the proximal method of multipliers. Conditions are provided which guarantee linear convergence.

In this section, we discuss strong convergence of (x_n) generated by the following algorithm which was studied by **Yao and Noor [12]** under different sets of assumptions on (α_n), (β_n), and (λ_n). Given any fixed $u, x_0 \in H$, the sequence (x_n) is generated by

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \gamma_n j_{\beta_n} x_n + e_n, \quad n \geq 0 \tag{1}$$

Where $\alpha_n \in (0,1), \lambda_n, \gamma_n \in (o,1)$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \geq 0$, $\beta_n \in (0, \infty)$, and (e_n) is a sequence of computational errors.

Theorem

Let $\beta_n \in (0, \infty)$, $\alpha_n \in (0, 1)$, and $\lambda_n, \gamma_n \in [0, 1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \geq 0$. Assume that $\mathbf{A: D(A) \subset H} \rightarrow \mathbf{2^H}$ is a maximal monotone operator with $F : A^{-1}(0) \neq \phi$, and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\dfrac{\|e_n\|}{\alpha_n}$ is bounded. Then for any fixed $u, x_0 \in H$, the sequence (x_n) defined by (1.1) is bounded.

Proof

If $\sum_{n=0}^{\infty} \|e_n\| < \infty$ · then one shows by induction that, for any $p \in F$ and $n \geq 0$,

$$\left\| x_n - p \right\| \leq \max \left(\left\| x_0 - p \right\|, \left\| u - p \right\| \right) + \sum_{n=0}^{n-1} \left\| e_k \right\| \tag{2}$$

Hence (x_n) is bounded.

Now assume that ($\left\| e_n \right\| / \alpha_n$) is bounded. Then, there exists a positive constant M such that

$$\left\| u - p \right\| + \frac{\left\| e_n \right\|}{\alpha_n} \leq M \tag{3}$$

for any $p \in F$ and all $n \geq 0$. Without loss of generality, we assume such a constant is such that $\| x_0 - p \| \leq 2M = $ **C**. We show by induction that, for all $n \geq 0$,

$$\left\| x_n - p \right\| \leq C \tag{4}$$

Using (1.1), and the sub-differential inequality, we have

$$\left\| x_{n+1} - p \right\|^2 = \left\| \alpha_n \left(u - p + \frac{e_n}{\alpha_n} \right) + \lambda_n \left(x_n - p \right) + \gamma_n \left(j_{\beta_n} x_n - p \right) \right\|^2 \tag{5}$$

$$\leq \left\| \lambda_n \left(x_n - p \right) + \gamma_n \left(j_{\beta_n} x_n - p \right) \right\|^2 + 2\alpha_n \left(\frac{u - p + \dfrac{e_n}{\alpha_n}}{x_{n+1} - p} \right) \tag{6}$$

$$\leq \left(1 - \alpha_n \right)^2 \left\| x_n - p \right\|^2 + 2M\alpha_n \left\| x_{n+1} - p \right\| \tag{7}$$

If $\| x_n - p \| \leq C$ for some $n \geq 0$, then the last estimate gives

$$\left\| x_{n+1} - p \right\|^2 \leq \left(1 - \alpha_n \right)^2 C^2 + 2M\alpha_n \left\| x_{n+1} - p \right\| \tag{8}$$

Hence,

$$\left(\left\| x_{n+1} - p \right\| - M\alpha_n \right)^2 \leq M^2 \alpha^2_n + \left(1 - \alpha_n \right)^2 C^2 \tag{9}$$

Which yields

$$\left\| x_{n+1} - p \right\| \leq M\alpha_n + \sqrt{M^2 \alpha^2_n + \left(1 - \alpha_n \right)^2 C^2} \tag{10}$$

Since the inequality

$$M\alpha_n + \sqrt{M^2 \alpha^2_n + \left(1 - \alpha_n \right)^2 C^2} \leq C \tag{11}$$

holds true, we conclude that (x_n) is bounded.

Theorem

Assume that $\mathbf{A} : \mathbf{D(A)} \subset \mathbf{H} \to 2^{\mathbf{H}}$ is a maximal monotone operator and $\boldsymbol{F} = \boldsymbol{A}^{-1}(\mathbf{0}) \neq \phi$ Fix $\boldsymbol{u}, \boldsymbol{x_0} \in$ H, and let

$(\boldsymbol{x_n})$ be the sequence generated by algorithm (1.1) with the conditions: **(i)** $\boldsymbol{\alpha_n} \in (0, 1)$, $\lim\limits_{n \to \infty} \alpha_n = \mathbf{0}$, **&** $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum\limits_{n=0}^{\infty} \|e_n\| < \infty$, $\quad or \quad \lim\limits_{n \to \infty} \dfrac{\|e_n\|}{\alpha_n} = \mathbf{0}$, **(iii)** $\lambda_{\mathbf{n}}, \gamma_{\mathbf{n}} \in [0, 1]$ $\alpha_n + \lambda_n + \gamma_n = \mathbf{1}$ with $\sum\limits_{n=0}^{\infty} \lambda_n < \infty$ and **(iv)** $\boldsymbol{\beta}\mathbf{n} \in (0,$

∞) with $\liminf\limits_{n \to \infty} \beta_n > \mathbf{0}$ and $\lim\limits_{n \to \infty} \left(\dfrac{1}{\beta_n} - \dfrac{1}{\beta_{n-1}} \right) = \mathbf{0}$ being satisfied. If, in addition $\sum\limits_{n=1}^{\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\beta_n} < \infty$ holds, then

$(\boldsymbol{x_n})$ converges strongly to $\boldsymbol{P_F} \, \boldsymbol{u}$, the projection of $\boldsymbol{u}$ on $\boldsymbol{F}$.

Proof

We have

$$v_n = \frac{x_{n+1} - \alpha_n u - \lambda_n x_n - e_n}{\gamma_n} \tag{12}$$

Note that $(\boldsymbol{v_n})$ is bounded since $(\boldsymbol{x_n})$ is bounded and for $\boldsymbol{\alpha_n}, \boldsymbol{\lambda_n} \to \mathbf{0}$, we see that the weak $\boldsymbol{\omega}$-limit sets of $(\boldsymbol{x_n})$ and $(\boldsymbol{v_n})$ coincide, that is, $\omega_w((x_n)) = \omega_w((v_n))$. Moreover, we have from (1.1), and (1.3),

$$Av_n \ni \frac{x_n - x_{n+1} - \alpha_n (u - x_n) + e_n}{\gamma_n} \tag{13}$$

Our aim is to prove that the relation $\omega_w((x_n)) \subset F$ holds, from which we can establish the inequality

$$\limsup_{n \to \infty} (u - P_F u, x_n - P_F u) \le \mathbf{0} \tag{14}$$

Indeed, for some subsequence $\left(x_{n_k} \right) of \left(x_n \right)$ converging weakly to some $x_\infty \in F$, we have

$$\limsup_{n \to \infty} (u - P_F u, x_n - P_F u) = \lim_{k \to \infty} (u - P_F u, x_{nk} - P_F u) = (u - P_F u, x_\infty - P_F u) \le \mathbf{0} \tag{15}$$

In view of (1.4), it would be enough if we could show that

$$\lim_{n \to \infty} \frac{x_{n+1} - x_n}{\beta_n} = \mathbf{0} \tag{16}$$

For this purpose, we first compare $\boldsymbol{x_{n+2}}$ and $\boldsymbol{x_{n+1}}$ as follows

$$x_{n+2} - x_{n+1} = (1 - \alpha_n)\left(J_{\beta_{n+1}} x_{n+1} - J\beta_n x_n \right) + \lambda_{n+1}\left(x_{n+1} - J_{\beta_{n+1}} x_{n+1} \right) \tag{17}$$

$$+ \lambda_n \left(J_\beta x_n - x_n \right) + (\alpha_{n+1} - \alpha_n)\left(u - J_{\beta_{n+1}} x_{n+1} + \frac{e_{n+1}}{\alpha_{n+1}} \right) + \alpha_n \left(\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right) \tag{18}$$

Using the resolvent identity and the fact that the resolvent operator is non-expansive, we get

$$\left\| x_{n+2} - x_{n+1} \right\| \le \left(1 - \alpha_n \right) \left\| J_{\beta_{n+1}} x_{n+1} - J_{\beta_{n+1}} \left(\frac{\beta_{n+1}}{\beta_n} x_n + \left(1 - \frac{\beta_{n+1}}{\beta_n} \right) J_\beta x_n \right) \right\|$$

(19)

$$+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| + \left(\lambda_n + \lambda_{n+1} \right) k + \left(\alpha_{n+1} - \alpha_n \right) M$$

(20)

$$\le \left(1 - \alpha_n \right) \left\| \frac{\beta_{n+1}}{\beta_n} \left(x_{n+1} - x_n \right) + \left(1 - \frac{\beta_{n+1}}{\beta_n} \right) \left(\left(x_{n+1} - J_{\beta n} x_n \right) \right) \right\|$$

(21)

$$+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| + \left(\lambda_n + \lambda_{n+1} \right) K + \left| \alpha_{n+1} - J_{\beta n} x_n \right| M$$

(22)

$$\le \left(1 - \alpha_n \right) \frac{\beta_{n+1}}{\beta_n} \left(\left\| x_{n+1} - x_n \right\| \right) + \left| 1 - \frac{\beta_{n+1}}{\beta_n} \right| \left\| x_{n+1} - J_{\beta n} x_n \right\|$$

(23)

$$+ \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| + \left(\lambda_n + \lambda_{n+1} \right) K + \left| \alpha_{n+1} - J_{\beta n} x_n \right| M \quad (24)$$

for some positive constants K and M. Note that we have from (1.1)

$$\left\| x_{n+1} - J_{\beta n} x_n \right\| \le \alpha_n \left\| u - J_{\beta_n} x_n + \frac{e_n}{\alpha_n} \right\| + \lambda_n \left\| x_n - J_{\beta n} x_n \right\|$$

(24)

which together with (1.7) yields

$$\frac{\left\| x_{n+2} - x_{n+1} \right\|}{\beta_{n+1}} \le \left(1 - \alpha_n \right) \frac{x_{n+1} - x_n}{\beta_n} + \alpha_n \left| \frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right| M + \frac{\alpha_n}{\beta_{n+1}} \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| + \frac{\left| \alpha_{n+1} - \alpha_n \right|}{\beta_{n+1}} M + \left(2\lambda_n + \lambda_{n+1} \right) K'$$

(25)

for some $K' > 0$.

The application of the sub-differential inequality to (1.1) yields

$$\left\| x_{n+1} - P_{Fu} \right\|^2 \le \left(1 - \alpha_n \right) \left\| x_n - P_{Fu} \right\|^2 + 2\alpha_n \left(u - P_{Fu} + \frac{e_n}{\alpha_n}, x_{n+1} - P_{Fu} \right)$$

(26)

Remark

Since λ_n is summable and (x_n) is bounded, the term $\lambda_n (x_n - J_{\beta n} x_n)$ can be regarded as the error term in the case when the sequence of errors is also summable.

CONCLUSIONS

We have analyzed the convergence of exponential multiplier method with multiple parameters by the theorems. These theorems are related to the proximal point algorithm that of strong convergence and that of acceptable errors with Yao and Noor's algorithm.

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